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ON THE STABILITY OF ROTATING, AXIALLY LOADED, HOMOGENEOUS SHAFTS

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Abstract—Using the example of the rotating Timoshenko-beam subjected to axial pressure load with internal and external damping a general circulatory vibration system with distributed parameters is formulated, in which stability behaviour is discussed in detail.

In particular the effect of gyroscopic stabilization and its influence by the different damping mechanisms is studied. By the means of modern operator methods the well-known theorems of Thomson and Tait, for instance, can be generalized for one-dimensional, continuous rotor systems.

1. INTRODUCTION

The equations of stability are well-known for very general rotor systems with concentrated parameters, which can be described in the usual matrix form

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{D} + \mathbf{G})\dot{\mathbf{q}} + (\mathbf{K} + \mathbf{N})\mathbf{q} = \mathbf{0}, \qquad (1)$$

where dots denote derivatives with respect to time t. For a mechanical n-degree of freedom-system $\mathbf{q}(t)$ is an n-dimensional representation vector, $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ are symmetric and $\mathbf{G}$ and $\mathbf{N}$ are antisymmetric matrices of order n. The mass-matrix $\mathbf{M}$ is always positive definite. A detailed discussion of the stability equations according to Lyapunov's theory of stability can be found, e.g. in [1]. There it is also shown, that the stability behavior of discrete, rotationally symmetrical, gyroscopic systems with internal and external damping is a general unsolved problem, which can be solved only by a quantitative calculation for every individual case. Certain necessary and sufficient conditions for stability, which exist[1, 2], can namely not be utilized. On the other hand, by neglecting different influences in a stepwise manner, this complex dynamic system allows the transition to simpler vibrating structures in a characteristic way. The properties of stability can be then judged by global theorems and general statements, based on modern matrix methods. For example, according to Thomson and Tait's theorems, for a so-called **M-D-G-K**-system, e.g. an axially loaded rotor influenced only by a pervasive damping, a gyroscopic stabilization of the statically unstable system ($\mathbf{K} < 0$) is impossible.

On the contrary, for elastic rotors with distributed parameters there has existed till now neither the formulation of an analogous general continuum model nor with exception of a paper by Shieh[3] corresponding statements of stability, which prove stability or instability by means of operator methods[4], without taking the usual path by a splitting into discrete parts of the continuum[2].

This is the starting point of this paper as in [5, 6]. With the aid of one-dimensional continua, such considerations will be extended to distributed, dynamic systems, which, incidentally, is the aim of the investigations.

2. ROTOR MODEL AND EQUATIONS OF STABILITY

It appears, that the rotating Timoshenko-beam under axial pressure load in connection with internal and external damping, as in Fig. 1, represents a general gyroscopic system in the above sense.

In particular, a rod-shaped, circular, elastic solid body with the length l, which rotates in stationary operation with a constant angular velocity ω_0 has been taken into consideration. It has a bending rigidity EI, a shearing rigidity κGA, mass per unit length μ and a radius of gyration k_r. For the sake of simplification, all parameters may be constant. The column should be simply supported on its two ends and subjected to a conservative, time-independent compressive load F. Because the boundary of stability reacts extremely sensitively to trace-

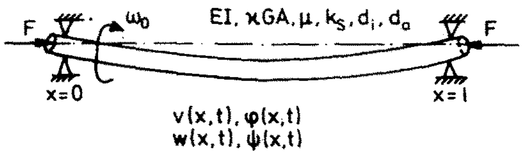

Fig. 1. Rotor model.

effects, it is necessary to include damping influences. Besides the usual, external damping, which is set proportional to the absolute velocity of the shaft-center in the form

$$\mathbf{F}_{Da} = -d_a \mathbf{v}_{abs},$$

internal damping should be considered, too. Realistically, the relation between the stress and strain tensors has to be modified in the sense of a viscoelastic law of material. Academically speaking, a viscous, internal damping, which is in the form

$$\mathbf{F}_{Di} = -d_i \mathbf{v}_{rel}$$

proportional to the relative velocity of the shaft-center in a rotating coordinate system, in analogy to the external damping, should be working here. As present investigations show, this straightforward damping formulation is justified, because the results of both internal damping mechanisms differ only quantitatively.

If the small bending vibrations of the rotor described by the position- and time-dependent transverse displacements $v(x, t)$ and $w(x, t)$, as well as the angles of inclination $\varphi(x, t)$ and $\psi(x, t)$-all of them measured in an inertial coordinate system-, one of the well-known principles of mechanics yields the appertaining boundary value problem

$$\left.\begin{aligned} &\mu v_{tt} + d_a v_t + d_i(v_t + \omega_0 w) - \kappa GA(v_{xx} - \varphi_x) + F v_{xx} = 0, \\ &\mu w_{tt} + d_a w_t + d_i(w_t - \omega_0 v) - \kappa GA(w_{xx} + \psi_x) + F w_{xx} = 0, \\ &\mu k_s^2 \varphi_{tt} - 2\omega_0 \mu k_s^2 \psi_t - EI\varphi_{xx} - \kappa GA(v_x - \varphi) = 0, \\ &\mu k_s^2 \psi_{tt} + 2\omega_0 \mu k_s^2 \varphi_t - EI\psi_{xx} + \kappa GA(w_x - \psi) = 0, \end{aligned}\right\} \tag{2.1}$$

$$v(j) = w(j) = \varphi_x(j) = \psi_x(j) = 0, \quad j = 0, l. \tag{2.2}$$

The subscripts t and x are derivatives with respect to time and position. According to D'Alembert, all relations, eqn (2), can be vividly interpreted as equilibriums of forces and momentums.

The boundary value problem, eqn (2), may be written in the alternate form [5, 6]

$$\mathbf{M}[\mathbf{q}]_{tt} + (\mathbf{D} + \mathbf{G})[\mathbf{q}]_t + (\mathbf{K} + \mathbf{N})[\mathbf{q}] = \mathbf{0}, \tag{3.1}$$

$$[\mathbf{q}(j)]^T \cdot \{\mathbf{M}_j[\mathbf{q}(j)]_{tt} + (\mathbf{D}_j + \mathbf{G}_j)[\mathbf{q}(j)]_t + (\mathbf{K}_j + \mathbf{N}_j)[q(j)]\} = \mathbf{0}, \tag{3.2}$$

where $\mathbf{q}(x, t)$ is a multidimensional representation vector in a functional space and $\mathbf{M}$, $\mathbf{D}$, $\mathbf{G}$, $\mathbf{K}$, $\mathbf{N}$, $\mathbf{M}_j$, $\mathbf{D}_j$, $\mathbf{G}_j$, $\mathbf{K}_j$ and $\mathbf{N}_j$ ($j = 0, l$) are suitable, time-independent matrix differential operators. For the example of a rotating Timoshenko-beam, the vector $\mathbf{q}$ contains in the form

$$\mathbf{q} = [v, w, \varphi, \psi]^T \tag{4.1}$$

the displacements v and w and the angles ψ and ψ, while the matrices **M** to $\mathbf{N}_j$ are declared as

$$\mathbf{M}=\begin{bmatrix}\mu & 0 & 0 & 0\\ 0 & \mu & 0 & 0\\ 0 & 0 & \mu k_s^2 & 0\\ 0 & 0 & 0 & \mu k_s^2\end{bmatrix},\quad \mathbf{D}=\begin{bmatrix}d_a+d_i & 0 & 0 & 0\\ 0 & d_a+d_i & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\mathbf{G}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -2\omega_0\mu k_s^2\\ 0 & 0 & +2\omega_0\mu k_s^2 & 0\end{bmatrix},$$

$$\mathbf{K}=\begin{bmatrix}(F-\kappa GA)(\cdots)_{xx} & 0 & \kappa GA(\cdots)_x & 0\\ 0 & (F-\kappa GA)(\cdots)_{xx} & 0 & -\kappa GA(\cdots)_x\\ -\kappa GA(\cdots)_x & 0 & -EI(\cdots)_{xx}+\kappa GA & 0\\ 0 & \kappa GA(\cdots)_x & 0 & -EI(\cdots)_{xx}+\kappa GA\end{bmatrix}$$

$$\mathbf{N}=\begin{bmatrix}0 & d_i\omega_0 & 0 & 0\\ -d_i\omega_0 & 0 & 0 & \\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},\quad \begin{matrix}\mathbf{M}_j=\mathbf{D}_j=G_j\\ =\mathbf{N}_j=\mathbf{0},\end{matrix}\quad \mathbf{K}_j=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & (\cdots)_x & 0\\ 0 & 0 & 0 & (\cdots)_x\end{bmatrix},\quad j=0,l. \tag{4.2}$$

The representation according to eqn (3) is entirely equivalent to the well-known description of discrete, dynamic systems, as in eqn (1) and can be interpreted as equations of stability of a general circulatory, distributed system.

3. INSTABILITY CONDITIONS

Practical statements of stability do not exist, however, so that first of all only an evaluation in detail is considered. For this calculation the manner of writing in operator form—eqn (3)—is certainly of no use, so that appropriately one has to go back to the initial eqn (2).

Upon introducing the complex coordinates

$$z=v+iw,\quad \chi=\varphi-i\psi,\quad i=\sqrt{-1}, \tag{5}$$

as well as the dimensionless variables

$$\xi=\frac{x}{l},\quad \tau=\omega_1 t\left(\omega_1{}^2=\frac{EI}{\mu l^4}\right) \tag{6}$$

and parameters

$$r=\left(\frac{k_s}{l}\right)^2,\ s=\frac{EI}{\kappa GAl^2},\ f=\frac{Fl^2}{EI},\ \Omega=\frac{\omega_0}{\omega_1},\ \delta_{i,a}=\frac{d_{i,a}}{2\mu\omega_1} \tag{7}$$

a boundary value problem

$$A\ddot{\bar{z}}+B\bar{z}+C\ddot{z}+D\dot{z}+Ez=0,$$

$$
\begin{aligned}
&A = rs,\ B = 2rs(\delta_a + \delta_i - i\Omega),\\
&C = [1 - 2irs\Omega(2\delta_a + 3\delta_i)] - [s + r(1 - fs)]\frac{\partial^2}{\partial\xi^2},\\
&D = 2\left\{(\delta_a + \delta_i - 2rs\Omega^2\delta_i) - [s(\delta_a + \delta_i) - ir\Omega(1 - fs)]\frac{\partial^2}{\partial\xi^2}\right\},\\
&E = (1 - fs)\frac{\partial^4}{\partial\xi^4} + (f + 2is\Omega\delta_i)\frac{\partial^2}{\partial\xi^2} - 2i\Omega\delta_i
\end{aligned}
\tag{8}
$$

for the transverse displacement z alone is obtained, when the angle of inclination χ is eliminated. Dots above the variable z denote derivatives with respect to dimensionless time τ.

In order to use the classical stability theory of Lyapunov, a class of solutions, derived from the perturbation eqn (5), which can be approximated by means of convergent modal expansions in a series, should always be singled out. For one-dimensional continua it is possible without restrictions. For this purpose, the transverse displacement z (ξ, τ) is separated into a function of position and one of time in the form

$$z(\xi, \tau) = Z(\xi)\, e^{\lambda\tau}. \tag{9}$$

Putting this into the boundary value problem, eqn (8), yields the pertinent eigenvalue problem

$$
\begin{aligned}
&aZ'''' + bZ'' + cZ = 0,\quad Z(j) = Z''(j) = 0,\quad j = 0, 1,\\
&a = 1 - fs,\ b = -[s + r(1 - fs)]\lambda^2 - 2[s(\delta_a + \delta_i) + ir\Omega(1 - fs)]\lambda + (f + 2is\Omega\delta_i),\\
&c = rs\lambda^4 + 2rs(\delta_a + \delta_i - i\Omega)\lambda^3 + [1 - 2irs\Omega(2\delta_a + 3\delta_i)]\lambda^2 + 2(\delta_a + \delta_i - 2rs\Omega^2\delta_i)\lambda - 2i\Omega\delta_i
\end{aligned}
\tag{10}
$$

for the generally complex eigenvalue λ, where the dashes mark derivatives with respect to ξ.

The real part of the eigenvalues determines stability or instability. If only the real part of a single eigenvalue becomes positive, the general solution $z(\xi, \tau)$, as in eqn (9), contains terms, which increase without limit, so that the axial compression of the rotor as the fundamental state, for which stability should be investigated, becomes unstable. Of particular interest is the dependence of the eigenvalues on the loading parameter f and speed of rotation Ω with as further parameters, the column-data r, s and the measures of damping $\delta_{i,a}$. When the real part of the eigenvalues, e.g. changes from initially negative to positive values, because the load has varied, there is the buckling force as a boundary of the region of stability. On the other hand, when the load is unchanged and the angular velocity increases, a so-called critical limiting-speed of rotation is reached.

The eigenvalue problem, eqn (10), holds in the form of the factors a, b, c evidently position-independent coefficients, so that a rigorous calculation of the eigenvalue equation is ensured.

The exponential form

$$Z(\xi) = \sum_{j=1}^{4} \alpha_j\, e^{\nu_j\xi} \tag{11}$$

as a general solution of the differential equation (10), whereby the four roots ν_j have to be evaluated from the corresponding characteristic equation

$$a\nu^4 + b\nu^2 + c = 0, \tag{12}$$

gives, after fitting to the boundary conditions, eqn (10), a homogeneous, algebraic equation system for the constants α_j. As a necessary condition for non-trivial solutions the pertinent coefficient determinant must vanish, and this,

$$\Delta[\lambda(\nu_j)] = 0, \tag{13}$$

is exactly the valid eigenvalue equation, which is evaluated simultaneously with the characteristic eqn (12).

Owing to the simple boundary values, eqn (10), especially here, the eigenform parameter ν_l can be determined beforehand independently from λ. Finally, the explicit, algebraic equation of condition

$$\begin{aligned} &rs\lambda^4 + 2rs(\delta_a + \delta_i - i\Omega)\lambda^3 + \{(n\pi)^2[r(1-fs)+s]+1 \\ &-2irs\Omega(2\delta_a + 3\delta_i)\}\lambda^2 + 2\{[1+s(n\pi)^2](\delta_a+\delta_i) \\ &-ir\Omega(1-fs)(n\pi)^2 - 2rs\Omega^2\delta_i\}\lambda + \{(1-fs)(n\pi)^4 \\ &-f(n\pi)^2 - 2i\Omega\delta_i[1+(n\pi)^2]\} = 0, \quad n = 1,2,\ldots \end{aligned} \tag{14}$$

is obtained for the four-fold infinitely many eigenvalues λ_{n1} to λ_{n4}.

The derivated eigenvalue equation (14) of the rotating, axially loaded Timoshenko-beam which is simply supported, corresponds to the lowest order of eigenvalue $n = 1$, which De Pater [7] has found for a discrete rotor system. Moreover, a range of special cases is included. Neglecting the shearing deformation by $s = 0$, e.g. leads from the Timoshenko-beam to the model of the so-called Rayleigh-bar. If the terms of rotational inertia with reference to the bending deformations are also suppressed, then $r = 0$ is valid as well, and the simplest bar-model, Euler's column with its sufficiently known eigenvalue equation is the result.

Now I would like to go back again to the general eigenvalue equation (14). Mostly, its solutions are computed numerically, but for the study of damping influences, also by means of the perturbation method [8, 9]. The essential results are noted in Figs. 2–4, where, the first, the case without damping $\delta_{i,a} = 0$ should be discussed.

Because the transition to unstable solutions regulary ensues in a way, that from two previously completely different pure imaginary eigenvalues of the order n a complex pair of eigenvalues will be produced with a coincident imaginary part and correspondingly equally large real part, one can limit oneself to a representation of the imaginary part of the eigenvalues as a function of the angular velocity Ω and the axial load f.

The first graph in Fig. 2 shows the variation as a function of the pressure load f for different rotor speeds Ω_k, while the second diagram illustrates the dependence of the angular velocity Ω on several values of the load f_k. The nth critical load can then be clearly recognized in a way, that the imaginary part of the eigenvalues λ_{n1} and λ_{n2} assume coincident values. When these critical loads of the order n and especially the buckling force for $n = 1$ are then plotted as a function of the rotor speed, the gyroscopic stabilization of the undamped rotating vibration

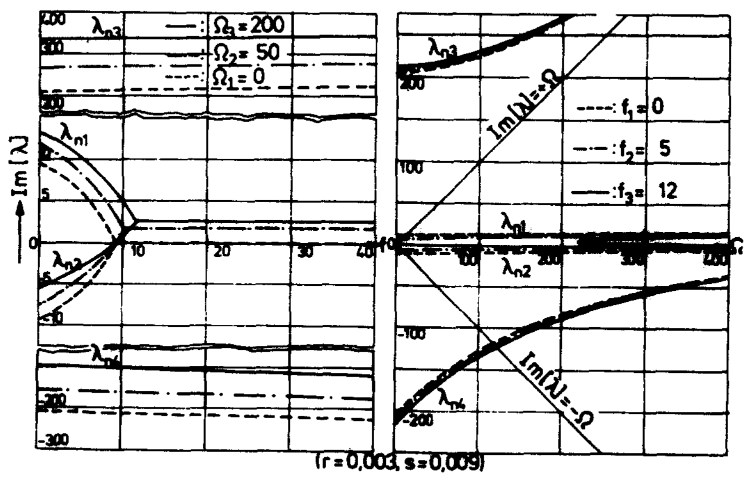

Fig. 2. Imaginary part of the eigenvalues λ as a function of velocity Ω and load f ($n = 1$, $\delta_{i,a} = 0$).

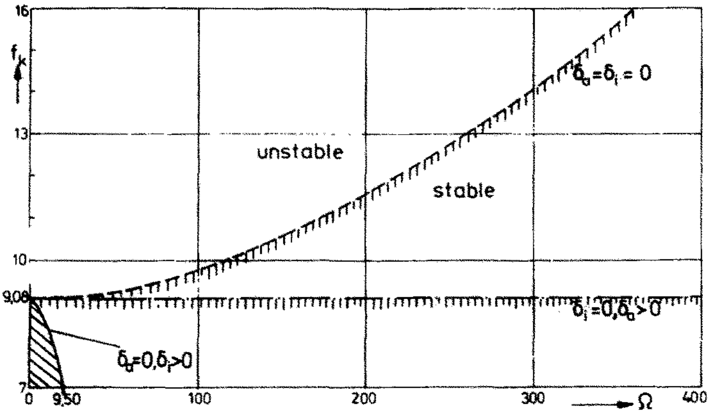

Fig. 3. Critical load f_k ($k = 1$) as a function of velocity Ω (δ_i or $\delta_a = 0$, $r = 0.003$, $s = 0.009$).

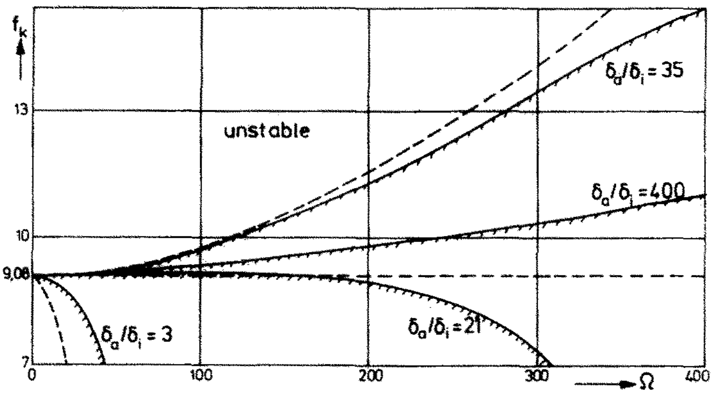

Fig. 4. Critical load f_k ($k = 1$) as a function of velocity (δ_i and $\delta_a \neq 0$, $r = 0.003$, $s = 0.009$).

system can be seen clearly, as in Fig. 3. Thus it is found, the buckling load increases with angular velocity Ω. All statements are insignificantly modified generalizations of De Pater's results [7] about a pressed rotor with a massless shaft and a fixed rigid disc.

It is well-known from discrete systems, that the gyroscopic stabilization has to be examined critically. According to the above mentioned theorems of Thomson and Tait it is extremely sensitive when neglecting influences of damping. The evaluation verifies this assertion in an impressive manner also for homogeneous shafts.

External damping cancels the gyroscopic stabilization again, Fig. 3, and the critical load is independent of the angular velocity Ω and identical to the static buckling force.

Internal damping becomes more destabilizing, Fig. 3 again, and the critical load decreases sharply with Ω. Above a definite limiting-speed of rotation even the unloaded Timoshenko-beam has an unstable rotation; an effect, which is well-known from classical rotor dynamics.

The influence of damping consequently operates in the same way as for a corresponding single-mass-rotor, investigated by De Pater [7]. Even there, qualitatively the same phenomenons appear, despite his remarks, which are incorrect.

If both the damping effects are acting—that is the situation in practice—a gyroscopic stabilization is possible, Fig. 4, once again, in fact, not only for a discrete rotor system, not dealt with by De Pater, but also for the Timoshenko-beam, which is discussed here. There is a specific ratio of external to internal damping, which leads to an increase in the boundary of stability till nearly the limit of the undamped case, for a system with concentrated parameters totally in agreement with the existing stability theory.

4. GENERAL STABILITY THEORY

Finally, it is shown that most results for distributed rotor systems can be predicted qualitatively without computation, as well. Using a general theory of stability of one-dimensional, circulatory vibration systems with distributed parameters, developed by Kelkel[5, 6] on the basis of Barston's investigations[4]—similar considerations are derived by Tasso[10]—general statements can be made. For this purpose, the real part of the eigenvalue λ is generally calculated. At this point the utility of the notation in operator form according to eqn (3) appears, which enables a clear and compact description of the following points.

A formulation

$$\mathbf{q}(x, t) = \mathbf{Q}(x)\, e^{\lambda t} \tag{15}$$

of the solution $\mathbf{q}(x, t)$, in the same way as before for the deformation $z(\xi, \tau)$, eqn (9), yields the eigenvalue problem

$$[\mathbf{M}\lambda^2 + (\mathbf{D} + \mathbf{G})\lambda + (\mathbf{K} + \mathbf{N})][\mathbf{Q}] \equiv \mathbf{L}[\mathbf{Q}] = \mathbf{0}, \tag{16.1}$$

$$[\mathbf{Q}(j)]^T \cdot \{[\mathbf{M}_j\lambda^2 + (\mathbf{D}_j + \mathbf{G}_j)\lambda + (\mathbf{K}_j + \mathbf{N}_j)][\mathbf{Q}(j)]\} = 0, \quad j = 0, l \tag{16.2}$$

also in operator form.

Considering the inner products

$$\begin{aligned}(\mathbf{Q}^{*T}, \mathbf{L}[\mathbf{Q}]) &= 0,\\ (\mathbf{Q}^T, \mathbf{L}^*[Q^*]) &= 0,\end{aligned} \tag{17}$$

where superscripts* denote the complex conjugate of Q or L, first adding and then subtracting the eqns (17) gives

$$\begin{aligned}k - \mathrm{Re}[\lambda]d - \mathrm{Im}[\lambda]\cdot g + (\mathrm{Re}^2[\lambda] - \mathrm{Im}[\lambda])\cdot m &= 0,\\ n + \mathrm{Im}[\lambda]\cdot d - \mathrm{Re}[\lambda]\cdot g - 2\,\mathrm{Re}[\lambda]\cdot \mathrm{Im}[\lambda]\cdot m &= 0,\end{aligned} \tag{18}$$

when the boundary conditions, eqn (16.2), are used. From the eqn (18) the real part of the eigenvalues may be found in the form

$$\mathrm{Re}[\lambda] = \frac{1}{2m}\left\{d \pm \frac{d^2 - g^2 - 4km}{2} + \frac{1}{2}\sqrt{[(d^2 - g^2 - 4km)^2 + 4(dg - mn)^2]}\right\}. \tag{19}$$

The quantities m, d, g, k and n denote characteristic functionals, which are, respectively, actions of inertia, damping and gyroscopic influences and, finally, conservative and circulatory restoring forces. But also for distributed parameter systems a unique classification is possible.

By a suitable definition of the definiteness of the functionals m, d, g, k and n a set of theorems can be derived, which decides stability or instability. But only for special cases of general circulatory systems are the statements useful in real situations.

For gyroscopic, conservative or general, noncirculatory systems, e.g. one obtains a generalization of Thomson and Tait's theorems. A conservative, gyroscopic $m-g-k$-system, the rotating, axially loaded Timoshenko-beam without damping effects, e.g. can be stable even for $k<0$, while if damping is to be added, a stabilization is impossible. Hence, a so-called $m-d-g-k$-system is asymptotically stable for d, $k>0$, whereas it is always unstable for $d \gtrless 0$ but $k \leq 0$. Most of the above mentioned, quantitatively calculated results have then been proved.

5. CONCLUDING REMARKS

The dynamic stability of distributed, mechanical systems raises many interesting questions. The example of rotating, axially loaded, homogeneous shafts is appropriate in a characteristic way to formulate and then to solve a stability problem of a rather general circulatory system with distributed parameters. It appears, that a great number of parallels to the corresponding stability problem of discrete vibration systems exists. Many results can be rediscovered, even if in modified form, and it can be supposed, that additional methods and abstractions can be transferred advantageously.

REFERENCES

1. P. C. Müller, *Stabilität und Matrizen.* Springer, Berlin (1977).
2. K. Huseyin, *Vibrations and Stability of Multiple Parameter Systems.* Nordhoff, Leyden (1978).
3. R. C. Shieh, Energy and variational principles for generalized (gyroscopic) conservative problems. *J. Nonlinear Mech.* **55**, 495 (1971).
4. E. M. Barston, Stability of dissipative systems. *Comm. Pure Appl. Math.* **22**, 627 (1969).
5. K. Kelkel, Stabilität dynamischer Systeme mit verteilten Parametern. Paper to presented at a *National Symp "Dynamische Systeme", Math. Forsch,* Inst. Oberwolfach (1979).
6. K. Kelkel, Kinetische Stabilität rotierender elastischer Stäbe. Habilitationsschrift Universität Karlsruhe (to be published).
7. A. D. De Pater, *The Motion of a Rotating Shaft Loaded by an Axial Force.* WTHD 59, Delft (1974).
8. M. Asseily, Zum Stabilitätsproblem des rotierenden Thimoshenko-Stabes unter konservativem Druck. Studienarbeit am Institut für Technische Mechanik, Universität Karlsruhe (non-published).
9. A. Schubert, Zum Einfluss innerer und äusserer Dämpfung auf das Schwingungsverhalten rotierender Wellen unter axialem Druck, Diplomarbeit am Institut für Technische Mechanik. Universität Karlsruhe (non-published).
10. H. Tasso, Dissipative MHD Stability, IPP 6/157 (1977).